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Conformal Models of Two–Dimensional Turbulence

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Polyakov recently showed how to use conformal field theory to describe two-dimensional turbulence. Here we construct an infinite hierarchy of solutions, both for the constant enstrophy flux cascade, and the constant energy flux cascade. We conclude with some speculations concerning the stability and physical meaning of these solutions.

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1. Introduction

Recently, Polyakov proposed a method for using conformal field theory to describe time averaged probability distributions in two-dimensional turbulence [1]. The essential idea was to use a local quantum field theory to describe correlation functions in the inertial range. One expects physics to be scale invariant in this range, so automatically the theory becomes conformally invariant [2].

Polyakov formulated the Navier-Stokes equations in terms of operators in the CFT, and used this to derive a matching condition that the effective CFT must satisfy. This followed from imposing the condition of constant enstrophy flux, which is the basis for the usual Kolmogorov-type arguments in two-dimensions [3]. In [1] one solution of the equations was found, namely the $(2,21)$ minimal model.

In this paper we construct an infinite series of solutions to the matching condition. In addition, solutions corresponding to a constant flux of energy are considered. This is known as the reverse cascade, since as shown in [3] energy flows from small scales to large scales. Again an infinite hierarchy of solutions seems to exist. We conclude with some speculations about the stability and physical meaning of these models.

2. Conformal field theory of turbulence

The Navier-Stokes equation in two-dimensions takes the form

$$\dot{\omega} + \epsilon_{\alpha\beta}\partial_\alpha\psi\partial_\beta\partial^2\psi = \nu\partial^2\omega \ , \tag{2.1}$$

where ω is the vorticity ($\omega = \partial^2\psi$) and ψ is the stream function, related to the velocity by $v_\alpha = -\epsilon_{\alpha\beta}\psi$. Here ν is the viscosity.

Polyakov's proposal is that correlators such as

$$\langle\omega(q_1)\cdots\omega(q_n)\rangle \ , \tag{2.2}$$

are described by conformal field theory when all the q_i and all sums over partitions of the q_i are in the inertial range: $1/R \ll q \ll 1/a$. Here R is the infrared cutoff, and a the ultraviolet cutoff, which is closely related to the viscosity. When considering correlators in position space, one should think, for example, of the stream function being decomposed as

$$\psi = \psi_c + \psi_{IR} \ , \tag{2.3}$$

where ψ_c is the conformal part, and ψ_{IR} is a piece that fluctuates only at IR scales.

Equation (2.1) is used to define the operator $\dot{\omega}$. For this to be well-defined, a point splitting regularization must be used. The result is that

$$\dot{\omega} \sim |a|^{2\Delta_\phi - 4\Delta_\psi}(L_{-2}\bar{L}_{-1}^2 - L_{-1}^2\bar{L}_{-2})\phi \ , \tag{2.4}$$

where ϕ is the minimal dimension operator that appears on the RHS of

$$\psi \times \psi \sim \phi + \cdots \ . \tag{2.5}$$

As we take the viscosity to zero, $a \to 0$, so we should require that $\dot{\omega}$ vanish in order that the inviscid Hopf equation be satisfied, i.e.

$$\langle \dot{\omega}(\vec{x}_1)\omega(\vec{x}_2)\cdots\omega(\vec{x}_n)\rangle + \langle \omega(\vec{x}_1)\dot{\omega}(\vec{x}_2)\cdots\omega(\vec{x}_n)\rangle + \cdots = 0 \ . \tag{2.6}$$

This requires the condition

$$\Delta_\phi > 2\Delta_\psi \ . \tag{2.7}$$

The condition that the effective conformal field theory match correctly with the enstrophy input at large scales and dissipated at small scales is that the enstrophy flux is constant. This is written as

$$\langle \dot{\omega}(r)\omega(0)\rangle = \text{const} \ , \tag{2.8}$$

i.e. is independent of R and r. Inserting the conformal fields one finds zero contribution. Polyakov then conjectures that the contributions arising from the IR part of the correlator must have a piece that scales with R in the way required by the anomalous dimensions of the conformal fields. We have then that

$$\langle \dot{\omega}(r)\omega(0)\rangle \sim |a|^{2\Delta_\phi - 4\Delta_\psi} R^{-2\Delta_\phi - 2\Delta_\psi - 6} r^0 \ . \tag{2.9}$$

Demanding this be independent of R leads to the condition

$$\Delta_\psi + \Delta_\phi = -3 \ . \tag{2.10}$$

Together with (2.7) this implies $\Delta_\psi < -1$ so the energy spectrum is always steeper than the Kolmogorov-Kraichnan result of $E(k) \sim k^{-3}$. Here one finds $E(k) \sim k^{4\Delta_\psi + 1}$.

In the following we will explore solutions of these equations, and also consider the case of constant energy flux, which may lead to an inertial range with a cascade of energy from small to large scales.

3. Minimal model solutions for constant enstrophy flux

Although the solutions of these equations for any conformal field theory are of interest, the minimal model solutions are a particularly interesting class, since only then are there a finite number of operators with negative dimension.

Before proceeding, let us review a few relevant facts about the minimal models [4] . The (p,q) minimal model (where $p<q$, with p and q relatively prime) contains $\frac{1}{2}(p-1)(q-1)$ degenerate primary operators which we label as $\psi_{m,n}$, where $1 \le n < q$ and $1 \le m < p$. These have conformal dimensions

$$\Delta_{m,n} = \frac{(pn-qm)^2-(p-q)^2}{4pq} \ , \tag{3.1}$$

and satisfy the fusion rules

$$\psi_{m_1,n_1} \times \psi_{m_2,n_2} = \sum_{i=|m_1-m_2|+1}^{\mathrm{Min}(m_1+m_2-1,2p-m_1-m_2-1)} \sum_{j=|n_1-n_2|+1}^{\mathrm{Min}(n_1+n_2-1,2q-n_1-n_2-1)} D^{(i,j)}_{(m_1,n_1)(m_2,n_2)} \psi_{i,j} \ , \tag{3.2}$$

where i runs over odd numbers if m_1+m_2-1 is odd, or even numbers if m_1+m_2-1 is even, and similarly for j. The $D^{(i,j)}_{(m_1,n_1)(m_2,n_2)}$ are the structure constants of the conformal field theory. Note there is a symmetry of the conformal dimensions $\Delta_{m,n} = \Delta_{p-m,q-n}$.

3.1. Solutions with $\psi = \psi_{1,s}$

Let us begin by classifying the solutions when $\psi = \psi_{1,s}$. Let $\phi = \psi_{1,t}$, and parametrize q by

$$q = tp + n \ . \tag{3.3}$$

Equation (2.10) then gives

$$p = \frac{2n(8-s-t)}{2-s^2-16t+2st+t^2} \ . \tag{3.4}$$

In order that, ϕ be the minimal dimension operator on the RHS of (2.5) we must have either

(a) $p\ \mathrm{Min}(2s-1, 2q-2s-1) < q$, or

(b) $|n| < p$.

Consider case (a). Set $t = 2s - 1$, since $t = 2q - 2s - 1$ is simply obtained by using invariance under $s \to q - s$. Substituting this into (3.4) we obtain

$$n = \frac{(19 - 38s + 7s^2)p}{9 - 3s} \ . \tag{3.5}$$

However for $s > 4$ this means $n < 0$ which is not possible, so it is only necessary to look for solutions with $s \in [1, 4]$. The only solution is the $(2, 21)$ model with $\psi = \psi_{1,4}$ and $\phi = \psi_{1,7}$ as found in [1].

Now consider case (b), $|n| < p$. The condition $\Delta_\psi < -1$ implies that $\Delta_\phi > -2$, so

$$\frac{(pt - q)^2 - (p - q)^2}{4pq} > -2 \ , \tag{3.6}$$

which leads to

$$p(12t - t^2 - 11) + 10 - 2t > 0 \ . \tag{3.7}$$

Since the fusion rules (3.2) imply t is an odd number, there are no solutions of this inequality for $t > 9$. It remains then to find the solutions for $t \leq 9$. There are a finite number of these as may be seen from the inequality derived from $|n| < p$,

$$\left|\frac{n}{p}\right| = \left|\frac{2 - s^2 - 16t + 2st + t^2}{2(8 - s - t)}\right| < 1 \ . \tag{3.8}$$

These solutions are displayed in appendix A, along with all other solutions for $q < 500$. Note that similar manipulations may be carried out whenever $\psi = \psi_{m,n}$ and for fixed m there will be a finite number of solutions. However m is unbounded so in general one expects an infinite number of solutions. This will be confirmed in the following. It should also be noted that the results of [5] tell us that solutions with spectra steeper than k^{-4} correspond to vanishing flux of enstrophy.

One situation in which ϕ need not be the minimal dimension operator on the RHS of (2.5) is when ϕ is degenerate on level 2, causing its contribution to $\dot{\omega}$ to vanish. The leading contribution to $\dot{\omega}$ would then come from the next to minimal operator on the RHS of (2.5) . It turns out, however, that this situation never arises in minimal model solutions.

3.2. Parametric solution of the general equations

For an arbitrary (p,q) minimal model solution with $\psi = \psi_{m,n}$ and $\phi = \psi_{s,t}$, equation (2.10) may be rewritten

$$r^2 + l^2 - 2(p^2 + q^2 - 8pq) = 0 \ , \tag{3.9}$$

where $l = (np - qm)$ and $r = (tp - qs)$. Defining $y = q - 4p$ this may be diagonalized

$$r^2 + l^2 + 30p^2 - 2y^2 = 0 \ . \tag{3.10}$$

This is a homogeneous quadratic Diophantine equation in four variables. It is well known [6] that the general solution to this equation in integers may be written as

$$\begin{aligned} r &= 2a^2 - 30b^2 - c^2 \\ l &= r + c(2c - 4a) \\ p &= |b(2c - 4a)| \\ y &= |r + a(2c - 4a)| \ , \end{aligned} \tag{3.11}$$

where a, b and c are integer parameters, and solutions differing by a constant are to be identified. Unfortunately it is difficult to implement the conditions $\Delta_\psi < -1$ and Δ_ϕ is minimal directly on this parametric solution, so many spurious solutions are generated. It is, however, a useful step in reducing the problem.

3.3. An infinite series of solutions

One interesting infinite sequence of solutions that do solve the constraints imposed by the OPE is the following:

$$\begin{aligned} \phi &= \psi_{1,9} \\ q &= 9p \pm 1 \\ l + (2p \pm 1)\sqrt{5} &= (9 \pm 4\sqrt{5})^j (1 \pm \sqrt{5}) \\ l &= np - mq \ , \end{aligned} \tag{3.12}$$

where $\psi = \psi_{m,n}$, and j is an integer parameter. Note that ψ is uniquely determined by solving the linear Diophantine equation for n and m. These solutions all satisfy

$$\Delta_\phi = \frac{1 - (p-q)^2}{4pq} \ , \tag{3.13}$$

so ϕ is the minimal dimension operator in the model. In addition it may be seen that ϕ appears in the OPE $\psi \times \psi$ for $j > 1$. As p and q become large, the critical exponent $4\Delta_\psi + 1 \to -3.89$.

3.4. Inequalities for general solutions

A stringent upper bound on the ratio p/q comes from equation (3.9) . Subtracting off r^2+l^2 one obtains

$$-2(p^2+q^2-8pq) \leq 0 \ , \tag{3.14}$$

so that

$$\frac{p}{q} < 4-\sqrt{15} \ . \tag{3.15}$$

One may also show that the $(2,21)$ model is the only model that does not satisfy $|r| < p$. This is done as follows. Suppose $|r| \geq p$. In order that ϕ be the minimal dimension operator that appears on the RHS of (2.5) we must have that $\phi = \psi_{1,t}$ and that $r = pt - q$ satisfy

$$p \leq r < q \ . \tag{3.16}$$

The parameter t will take its maximal value so that $|r|$ is minimized. For $\psi = \psi_{m,n}$ we have therefore that $t = 2n-1$. Equation (3.16) leads to

$$t < \frac{2q}{p} \ , \tag{3.17}$$

so that

$$n < \frac{q}{p} + \tfrac{1}{2} \ . \tag{3.18}$$

Now the condition $\Delta_\psi < -1$ can be rewritten as

$$m < \sqrt{(p/q)^2+1-6(p/q)} + 1 + \frac{p}{2q} \ . \tag{3.19}$$

Using (3.15) one finds that $m=1$ is the only possibility. Since all models with $m=1$ were classified above, one finds that the $(2,21)$ model is the only one that satisfies $|r| \geq p$.

This fact may be used to place a lower bound on p/q which is satisfied for all models other than the $(2,21)$ model. The inequality $\Delta_\psi < -1$ implies that $\Delta_\phi > -2$ so that (2.10) is satisfied. Using $|r| < p$ tells us

$$\frac{p^2-(p-q)^2}{4pq} > -2 \ , \tag{3.20}$$

which implies that

$$\frac{p}{q} > \frac{1}{10} \ , \tag{3.21}$$

for all solutions except the $(2,21)$ model.

These bounds on p/q then imply that the central charge

$$c = 1 - \frac{6(p-q)^2}{pq} \ , \tag{3.22}$$

is minimized for the $(2,21)$ model, where $c = -50\frac{4}{7}$. Unfortunately, the Zamolodchikov c-theorem [7] does not apply to conformal theories that do not satisfy reflection-positivity (which is always the case here). Otherwise one would then be able to argue that the $(2,21)$ model is at a fixed point under perturbations of the conformal field theory. The minimal value of c for the $(2,21)$ model does mean this model is stable under a large class of perturbations. Perhaps these are particularly relevant for hydrodynamic stability.

3.5. Parity properties

All (p,q) minimal models possess a Z_2 symmetry when one of p and q is odd, the other even. If one considers the whole Kac table $\psi_{m,n}$, with $1 \leq m < p$ and $1 \leq n < q$, and restricts consideration to operators with $n+m$ even, then operators with n and m odd are in the odd parity sector, and operators with n and m even are in the even parity sector.

We normally think of ψ as a pseudoscalar operator, so this should be in the odd parity sector. On the other hand, we know that $\phi = \psi_{t,s}$ has s and t both odd, so must also be in the odd parity sector. This leads one to conclude that there is no well defined parity for these solutions. In fact, ψ cannot be in the even parity sector either. This follows from the fact that $\Delta_\psi + \Delta_\phi$ then equals an odd number divided by an even number, so cannot satisfy (2.10) . It seems then that the minimal model solutions should break parity invariance. If correlation functions are symmetric under an additional Z_2 symmetry, this may then be used to define spatial parity, so in some cases, parity preserving models may exist.

4. Constant energy flux cascade

The constant energy flux inertial range may be considered in the same way as above. The Navier–Stokes equations are

$$\dot{v}_\alpha + v_\beta \partial_\beta v_\alpha = -\frac{1}{\rho}\partial_\alpha p + \nu \partial^2 v_\alpha \ . \tag{4.1}$$

Taking the divergence of this we get

$$\frac{1}{\rho}\partial^2 p = -\partial_\alpha\partial_\beta(v_\alpha v_\beta) \ , \tag{4.2}$$

determining the pressure in terms of the velocity field. Neglecting viscosity, and working in momentum space we find that

$$\dot{v}_\alpha(q) = -(v_\beta\partial_\beta v_\alpha)(q) + \frac{q_\alpha q_\beta}{q^2}(v_\gamma\partial_\gamma v_\beta)(q) \ . \tag{4.3}$$

As before, we define $v_\beta\partial_\beta v_\alpha$ using a point–split regularization, and use the operator product expansion to obtain

$$\begin{aligned} v_\beta\partial_\beta v_z(\vec{x}) &\sim |a|^{2\Delta_\phi-4\Delta_\psi-2}BL_{-1}\phi + |a|^{2\Delta_\phi-4\Delta_\psi}(L_{-2}\bar{L}_{-1} + AL_{-1}^2\bar{L}_{-1})\phi + \cdots \\ v_\beta\partial_\beta v_{\bar{z}}(\vec{x}) &\sim |a|^{2\Delta_\phi-4\Delta_\psi-2}B\bar{L}_{-1}\phi + |a|^{2\Delta_\phi-4\Delta_\psi}(\bar{L}_{-2}L_{-1} + A\bar{L}_{-1}^2L_{-1})\phi + \cdots \end{aligned} \tag{4.4}$$

where A and B are constants, determined by the operator product expansion. Note that by taking the curl of (4.4) we find agreement with (2.4) as desired. Inserting (4.4) into (4.3) we find that the piece proportional to B drops out, and so

$$\begin{aligned} \dot{v}_z(q) \sim |a|^{2\Delta_\phi-4\Delta_\psi}\Big(&-\big((L_{-2}\bar{L}_{-1} + AL_{-1}^2\bar{L}_{-1})\phi\big)(q) \\ &+ \frac{q_z}{q^2}\Big(q_z\big((\bar{L}_{-2}L_{-1} + A\bar{L}_{-1}^2L_{-1})\phi\big)(q) + q_{\bar{z}}\big((L_{-2}\bar{L}_{-1} + AL_{-1}^2\bar{L}_{-1})\phi\big)(q)\Big)\Big) \ . \end{aligned} \tag{4.5}$$

Now, following the same kind of arguments as lead to (2.10) , the condition of constant energy flux

$$\langle \dot{v}_\alpha(r)v_\alpha(0)\rangle \sim r^0 \ , \tag{4.6}$$

leads to the condition

$$\Delta_\psi + \Delta_\phi = -2 \ . \tag{4.7}$$

The inequality $\Delta_\phi > 2\Delta_\psi$, obtained by requiring the vanishing of (4.5) as $a \to 0$ means that

$$\Delta_\psi < -2/3, \qquad\qquad \Delta_\phi > -4/3 \ , \tag{4.8}$$

so the energy spectrum must be steeper than the Kolmogorov value.

5. Minimal model solutions for constant energy flux

Now, let us proceed to find minimal models that solve these conditions.

5.1. Solutions with $\psi = \psi_{1,s}$

The proof is the same as before. Let $\phi = \psi_{1,t}$ and parametrize q by $q = pt + n$. Equation (4.7) leads to

$$\frac{n}{p} = \frac{s^2 - t^2 + 12t - 2st - 2}{2(t + s - 6)} \ . \tag{5.1}$$

If $p \ \mathrm{Min}(2s-1, 2q-2s-1) < q$ then we must have that $t = 2s - 1$ (provided we shift $s \to q - s$ as appropriate). Substituting this into (5.1) one finds no solution for integer s with $n > 0$.

The solutions must therefore satisfy $|n| < p$. The inequality $\Delta_\phi > -4/3$ then gives

$$-t^2 + 28/3t - 25/3 > 0 \ . \tag{5.2}$$

Since t is odd this means $t \leq 7$. Again this leads to a finite number of solutions, which are displayed in appendix B, along with all other solutions for $q < 500$. Note that solutions with spectra steeper than $E(k) \sim k^{-8/3}$ must have vanishing energy flux, as shown in [5] .

5.2. Parametric solution of the general equation

For the (p, q) minimal model with $\psi = \psi_{m,n}$ and $\phi = \psi_{s,t}$ equation (4.7) becomes

$$r^2 + l^2 + 16p^2 - 2y^2 = 0 \ , \tag{5.3}$$

where $l = np - qm$, $r = tp - sq$ and $y = q - 3p$. The parametric solution of this equation is then

$$\begin{aligned} r &= 2a^2 - 16b^2 - c^2 \\ l &= r + 2c(c - 2a) \\ p &= |2b(c - 2a)| \\ y &= |r + 2a(c - 2a)| \end{aligned} \tag{5.4}$$

where a, b, c are integer parameters and solutions differing by a constant are to be identified. The additional conditions imposed by the OPE are difficult to impose on this solution so spurious solutions are still generated.

5.3. Inequalities

Equation (5.3) leads to the inequality

$$-2(p^2 + q^2 - 6pq) < 0 \ , \tag{5.5}$$

which means that

$$\frac{p}{q} < 3 - \sqrt{8} \ . \tag{5.6}$$

Following the same reasoning as before one may also prove that all solutions must satisfy $|r| < p$. Suppose $|r| \geq p$. In order that ϕ be minimal dimension we must have $|r| < q$. This leads to $t < 2q/p$ and $n < q/p + \frac{1}{2}$. Using the condition $\Delta_\psi < -2/3$ one finds that

$$m < \sqrt{1 + (\frac{p}{q})^2 - \frac{14p}{3q}} + 1 + \frac{p}{2q} \ . \tag{5.7}$$

Together with (5.6) this is only satisfied for $m = 1$. We have already determined that all solutions with $m = 1$ satisfy $|r| < p$, so this must be true in general.

This fact may then be used to find a lower bound on p/q by using $\Delta_\phi > -4/3$ which gives

$$\frac{p^2 - (p-q)^2}{4pq} > -\frac{4}{3} \ , \tag{5.8}$$

so that

$$\frac{p}{q} > \frac{3}{22} \ . \tag{5.9}$$

6. Stability?

Probably some of these solutions do not correspond to stable flow distributions. One would like to formulate an additional criterion, in terms of the effective conformal field theory description, that these solutions must satisfy. One essential condition that must be satisfied is that of reality of velocity correlation functions. This means velocity correlators must satisfy a Cauchy-Schwarz inequality. Unfortunately, it turns out to be impossible to neglect the IR sector of the theory in this type relation. For example, if one works in momentum space, one obtains relations such as

$$\langle \omega(p_1)\omega(p_2)|\omega(p_3)\rangle^2 = \langle \omega(p_1)\omega(p_2)|\omega(-p_1)\omega(-p_2)\rangle\langle \omega(p_3)|\omega(-p_3)\rangle \ , \tag{6.1}$$

which explicitly shows that a zero momentum intermediate state appears.

Perhaps then, all (or at least some) of these solutions do correspond to quasi-static equilibrium solutions. The picture that seems to emerge is that of an infinite hierarchy of different scaling behaviors, the stability of which depends strongly on the IR boundary conditions/ stirring forces.

There is actually some experimental evidence for this hypothesis. It is well known that in two-dimensional turbulence coherent vortex structures appear at the IR scale. The distribution of these seems to depend on the type of stirring forces used as in [8]. There it was found that when the stirring forces caused the coherent vortices to be unstable, the energy spectrum scaled as $E(k) \sim k^{-3.5}$ or $k^{-3.6}$, very close to the exact value for the $(2, 21)$ model of $E(k) \sim k^{-25/7}$. On the other hand, when the stirring forces favored coherent vortex formation, a scaling law of $E(k) \sim k^{-4.2}$ was found which happens to correspond closely to the third model shown in appendix A, the $(3, 25)$ model. It is also amusing to note that results from a lower resolution simulation quoted in [8] had a scaling law of $E(k) \sim k^{-4.7}$ which is close to that of the second model in appendix A.

It is natural to regard the $(2, 21)$ model as the simplest of the solutions to (2.10) since it has the lowest number of degenerate primary operators. It is not too surprising, then, that it should correspond to the simplest behavior of the fluid. As the distribution of the coherent vortices becomes more complicated, perhaps the whole hierarchy of models may be observed. Probably there is some kind of entropy principle that must be taken into account, which may restrict us, in practice, to observation of only the first few scaling behaviors.

7. Conclusion

An infinite sequence of models have been found which should describe turbulence in two dimensions. Perhaps the physical solutions must satisfy some additional condition which is necessary to correctly match the effective conformal field theory with the correct IR and UV behavior. If not, then an infinite hierarchy of inertial ranges seems likely, although perhaps only the first few of these may be observed in practice.

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Appendix A. Solutions of the constant enstrophy flux condition for $q < 500$

(p,q)	ψ	ϕ	$4\Delta_\psi + 1$	
(2,21)	$\psi_{1,4}$	$\psi_{1,7}$	$-25/7$	≈ -3.57
(3,25)	$\psi_{1,11}$	$\psi_{1,9}$	$-23/5$	≈ -4.60
(3,26)	$\psi_{1,5}$	$\psi_{1,9}$	$-55/13$	≈ -4.23
(6,55)	$\psi_{1,14}$	$\psi_{1,9}$	$-41/11$	≈ -3.73
(7,62)	$\psi_{1,13}$	$\psi_{1,9}$	$-125/31$	≈ -4.03
(8,67)	$\psi_{3,28}$	$\psi_{3,25}$	$-302/67$	≈ -4.51
(9,71)	$\psi_{4,32}$	$\psi_{7,55}$	$-173/35$	≈ -4.99
(11,87)	$\psi_{2,16}$	$\psi_{3,23}$	$-1605/319$	≈ -5.03
(11,91)	$\psi_{2,14}$	$\psi_{3,25}$	$-355/77$	≈ -4.61
(11,93)	$\psi_{2,20}$	$\psi_{3,25}$	$-1515/341$	≈ -4.44
(14,111)	$\psi_{1,8}$	$\psi_{1,7}$	$-187/37$	≈ -5.05
(14,115)	$\psi_{1,6}$	$\psi_{1,9}$	$-109/23$	≈ -4.74
(16,135)	$\psi_{7,56}$	$\psi_{7,59}$	$-40/9$	≈ -4.44
(21,166)	$\psi_{4,31}$	$\psi_{7,55}$	$-2895/581$	≈ -4.98
(22,179)	$\psi_{1,10}$	$\psi_{1,9}$	$-865/179$	≈ -4.83
(25,197)	$\psi_{11,87}$	$\psi_{9,71}$	$-4919/985$	≈ -4.99
(26,205)	$\psi_{5,39}$	$\psi_{9,71}$	$-2659/533$	≈ -4.99
(26,213)	$\psi_{9,76}$	$\psi_{5,41}$	$-4325/923$	≈ -4.69
(23,217)	$\psi_{6,62}$	$\psi_{9,85}$	$-2467/713$	≈ -3.46
(26,223)	$\psi_{1,12}$	$\psi_{1,9}$	$-965/223$	≈ -4.33
(27,229)	$\psi_{10,88}$	$\psi_{19,161}$	$-3025/687$	≈ -4.40
(25,234)	$\psi_{9,79}$	$\psi_{11,103}$	$-53/15$	≈ -3.53
(32,267)	$\psi_{11,89}$	$\psi_{3,25}$	$-1615/356$	≈ -4.54
(35,277)	$\psi_{14,110}$	$\psi_{11,87}$	$-9617/1939$	≈ -4.96

(p,q)	ψ	ϕ	$4\Delta_\psi + 1$	
(29,280)	$\psi_{15,139}$	$\psi_{3,29}$	$-94/29$	≈ -3.24
(33,287)	$\psi_{14,118}$	$\psi_{13,113}$	$-1889/451$	≈ -4.19
(33,299)	$\psi_{10,86}$	$\psi_{1,9}$	$-1145/299$	≈ -3.83
(34,311)	$\psi_{15,142}$	$\psi_{27,247}$	$-19793/5287$	≈ -3.74
(35,313)	$\psi_{6,58}$	$\psi_{1,9}$	$-1235/313$	≈ -3.95
(38,333)	$\psi_{13,110}$	$\psi_{17,149}$	$-235/57$	≈ -4.12
(34,335)	$\psi_{1,16}$	$\psi_{1,9}$	$-209/67$	≈ -3.12
(34,335)	$\psi_{15,154}$	$\psi_{7,69}$	$-3469/1139$	≈ -3.05
(39,346)	$\psi_{17,155}$	$\psi_{31,275}$	$-9031/2249$	≈ -4.02
(43,347)	$\psi_{11,89}$	$\psi_{15,121}$	$-71719/14921$	≈ -4.81
(47,378)	$\psi_{15,119}$	$\psi_{23,185}$	$-14311/2961$	≈ -4.83
(39,379)	$\psi_{14,142}$	$\psi_{25,243}$	$-1205/379$	≈ -3.18
(44,397)	$\psi_{21,194}$	$\psi_{1,9}$	$-1535/397$	≈ -3.87
(45,404)	$\psi_{19,175}$	$\psi_{1,9}$	$-395/101$	≈ -3.91
(48,413)	$\psi_{19,167}$	$\psi_{5,43}$	$-505/118$	≈ -4.28
(55,434)	$\psi_{14,111}$	$\psi_{9,71}$	$-1699/341$	≈ -4.98
(54,437)	$\psi_{23,188}$	$\psi_{11,89}$	$-18815/3933$	≈ -4.78
(59,486)	$\psi_{6,47}$	$\psi_{5,41}$	$-22201/4779$	≈ -4.65
(62,489)	$\psi_{13,103}$	$\psi_{9,71}$	$-25195/5053$	≈ -4.99
(61,497)	$\psi_{21,169}$	$\psi_{7,57}$	$-20485/4331$	≈ -4.73

Appendix B. Solutions of constant energy flux condition for $q < 500$

(p,q)	ψ	ϕ	$4\Delta_\psi + 1$	
(5,32)	$\psi_{1,9}$	$\psi_{1,7}$	$-5/2$	≈ -2.50
(6,37)	$\psi_{1,8}$	$\psi_{1,7}$	$-103/37$	≈ -2.78
(8,47)	$\psi_{3,17}$	$\psi_{5,29}$	$-140/47$	≈ -2.98
(9,58)	$\psi_{4,23}$	$\psi_{7,45}$	$-209/87$	≈ -2.40
(10,59)	$\psi_{1,6}$	$\psi_{1,5}$	$-181/59$	≈ -3.07
(10,63)	$\psi_{1,4}$	$\psi_{1,7}$	$-55/21$	≈ -2.62
(16,101)	$\psi_{5,34}$	$\psi_{3,19}$	$-511/202$	≈ -2.53
(16,105)	$\psi_{7,49}$	$\psi_{9,59}$	$-16/7$	≈ -2.29
(18,109)	$\psi_{7,44}$	$\psi_{13,79}$	$-913/327$	≈ -2.79
(22,147)	$\psi_{1,10}$	$\psi_{1,7}$	$-107/49$	≈ -2.18
(25,166)	$\psi_{9,63}$	$\psi_{11,73}$	$-917/415$	≈ -2.21
(32,187)	$\psi_{9,53}$	$\psi_{7,41}$	$-1117/374$	≈ -2.99
(31,192)	$\psi_{15,95}$	$\psi_{5,31}$	$-82/31$	≈ -2.65
(37,216)	$\psi_{8,47}$	$\psi_{7,41}$	$-997/333$	≈ -2.99
(36,241)	$\psi_{11,77}$	$\psi_{13,87}$	$-1559/723$	≈ -2.16
(37,248)	$\psi_{8,57}$	$\psi_{7,47}$	$-2465/1147$	≈ -2.15
(40,261)	$\psi_{19,121}$	$\psi_{21,137}$	$-202/87$	≈ -2.32
(44,269)	$\psi_{19,118}$	$\psi_{9,55}$	$-8057/2959$	≈ -2.72
(46,271)	$\psi_{17,101}$	$\psi_{9,53}$	$-18319/6233$	≈ -2.94
(45,274)	$\psi_{14,87}$	$\psi_{11,67}$	$-1129/411$	≈ -2.75
(47,274)	$\psi_{17,99}$	$\psi_{29,169}$	$-19313/6439$	≈ -3.00

(p,q)	ψ	ϕ	$4\Delta_\psi + 1$	
(51,320)	$\psi_{17,109}$	$\psi_{11,69}$	$-349/136$	≈ -2.57
(49,330)	$\psi_{13,91}$	$\psi_{15,101}$	$-163/77$	≈ -2.12
(58,339)	$\psi_{23,134}$	$\psi_{45,263}$	$-9779/3277$	≈ -2.98
(59,344)	$\psi_{6,35}$	$\psi_{5,29}$	$-7616/2537$	≈ -3.00
(59,364)	$\psi_{6,35}$	$\psi_{5,31}$	$-14347/5369$	≈ -2.67
(63,368)	$\psi_{4,23}$	$\psi_{7,41}$	$-1444/483$	≈ -2.99
(61,372)	$\psi_{8,47}$	$\psi_{11,67}$	$-5179/1891$	≈ -2.74
(62,377)	$\psi_{9,53}$	$\psi_{13,79}$	$-2477/899$	≈ -2.76
(62,385)	$\psi_{15,91}$	$\psi_{5,31}$	$-6277/2387$	≈ -2.63
(65,391)	$\psi_{31,185}$	$\psi_{61,367}$	$-623/221$	≈ -2.82
(66,395)	$\psi_{13,79}$	$\psi_{25,149}$	$-2531/869$	≈ -2.91
(67,396)	$\psi_{22,131}$	$\psi_{11,65}$	$-587/201$	≈ -2.92
(69,410)	$\psi_{32,189}$	$\psi_{17,101}$	$-2725/943$	≈ -2.89
(70,411)	$\psi_{21,124}$	$\psi_{39,229}$	$-2837/959$	≈ -2.96
(70,419)	$\psi_{11,67}$	$\psi_{21,125}$	$-8591/2933$	≈ -2.93
(72,431)	$\psi_{19,115}$	$\psi_{37,221}$	$-1244/431$	≈ -2.89
(64,433)	$\psi_{15,105}$	$\psi_{17,115}$	$-1807/866$	≈ -2.09
(64,435)	$\psi_{7,44}$	$\psi_{9,61}$	$-239/116$	≈ -2.06
(74,439)	$\psi_{31,185}$	$\psi_{15,89}$	$-47089/16243$	≈ -2.90
(68,453)	$\psi_{11,70}$	$\psi_{17,113}$	$-5641/2567$	≈ -2.20

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